The BFV Approach for a Nonlocal Symmetry of QED

Silvio J. Rabello* and Patricio Gaete†

Instituto de Física, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ

Caixa Postal 68.528-CEP 21945-970, Brasil

(February 7, 2008)

Abstract

In this paper we use the Batalin-Fradkin-Vilkovisky formalism to study a recently proposed nonlocal symmetry of QED. In the BFV extended phase space we show that this symmetry stems from a canonical transformation in the ghost sector.

PACS numbers: 12.20.-m,11.30.-j,11.15.-q

*e-mail: rabello@if.ufrj.br

†Present adress: Instituto de Física, Universidade Estadual do Rio de Janeiro, Rua São Francisco Xavier 524, Rio de Janeiro RJ, CEP 20550-013 Brasil

It is well known that gauge symmetry plays an essential role in the modern picture of quantum field theory, where the locality of the fields is implicit. Thus, in order to carry out a consistent quantization of a theory with gauge symmetry, it is necessary to eliminate the non-physical degrees of freedom. The usual procedure to achieve this is to impose auxiliary conditions (gauge conditions) and their form is dictated largely by computational convenience.

In the path integral quantization methods (both Lagrangian and Hamiltonian) the gauge invariance is included by means of the extension of the phase space incorporating the Grassmann odd ghost variables. In this case , the basic idea is to replace the local gauge symmetry by a global Becchi-Rouet-Stora-Tyutin supersymmetry (BRST) [1]. Today this BRST symmetry plays a crucial role in the quantization of gauge theories.

In this connection in recent times a great deal of attention has been devoted to the study of new symmetries of QED. Recently, Lavelle and McMullan [2] found that QED exhibits a new nonlocal and noncovariant graded symmetry, even so nilpotent. In such a case the symmetry transformations are compatible with the gauge fixing conditions. Moreover these authors claim that this new symmetry may be used to refine the characterization of the physical states given by the BRST charge. Following these lines, Tang and Finkelstein [3] constructed a covariant graded symmetry for QED of which the noncovariant symmetry of Lavelle and McMullan is a special case. Also Yang and Lee [4] derived a noncovariant but local symmetry of QED. The Noether charges for all these symmetries are nilpotent and impose constraints on the physical states. Nevertheless, these new symmetries reported up till now are all constructed from a Lagrangian point of view.

On the other hand, the Hamiltonian formalism developed by Batalin, Fradkin and Vilkovisky (BFV) [5] provides a powerful method for the BRST quantization of constrained systems, e.g., QED. As it is well known, some properties of the BFV approach are: it does not need an auxiliary field and the BRST transformations are independent of the gauge conditions, it uses an extended phase space in which the Lagrange multipliers and ghosts are introduced as dynamical variables. The BRST charge for first class constrained systems

can be constructed directly from the constraints. Thus, the algebraic structure of the constraints is captured in the BRST charge and its nilpotency in a gauge independent way, unlike the Lagrangian formulation in which the BRST charge is calculated from a gauge fixed Lagrangian via Noether's theorem.

It is therefore motivating in this context to consider the question of how arises, in the Hamiltonian approach, the Lavelle and McMullan's symmetry. In order words we want to find the relation between the Lagrangian and Hamiltonian symmetry generators. A detailed discussion of the BFV formalism can be found in [6]. We here collect some results of the BFV treatment for QED. The Lagrangian for QED with photon gauge field A_μ and Dirac electron field ψ is given by [1]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \,, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$. The canonical momenta of the gauge fields are $\pi_\mu = F_{0\mu}$ with the only nonvanishing canonical Poisson brackets being:

$$[A^\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = \delta^\mu_\nu \delta(\mathbf{x} - \mathbf{y}) \,, \tag{2}$$

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] = -i\delta(\mathbf{x} - \mathbf{y}) \tag{3}$$

As we can see there is one primary constraint, $\pi_0 = 0$, and the canonical Hamiltonian is

$$H = H_0 + \int d^3x A_0 (\nabla \cdot \boldsymbol{\pi} + e\psi^\dagger \psi) \,, \tag{4}$$

where

$$H_0 = \int d^3x \Big(\psi^\dagger (i\boldsymbol{\alpha} \cdot \mathbf{D} + \gamma_0 m)\psi + \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{4} F^{ij} F_{ij} \Big) \,. \tag{5}$$

The conservation in time of the constraint $\pi_0 = 0$ gives us the Gauss law:

$$\nabla \cdot \boldsymbol{\pi} + e\psi^\dagger \psi = 0 \tag{6}$$

---

[1] $\hbar = c = 1$ and $\eta_{\mu\nu} = diag(1, -1, -1, -1)$

There are no more constraints in the theory and the two we have found are first class (they have a vanishing Poisson bracket). Having identified the first class constraints we are ready to apply the BFV method where one starts with the Gauss law constraint in the phase space (A_i, π_i) and introduces the pair of anticommuting ghosts $(C(x), \mathcal{P}(x))$ of ghost number 1 and -1 respectively. Then one adds the Lagrange multiplier $A_0(x)$ and its vanishing momentum π_0, to take care of this new constraint we have the antighost pair $(\bar{C}(x), \bar{\mathcal{P}}(x))$ of ghost number -1 and 1 respectively. The Poisson algebra of these ghosts is (the nonzero part of it)

$$[C(\mathbf{x},t), \mathcal{P}(\mathbf{y},t)] = [\bar{C}(\mathbf{x},t), \bar{\mathcal{P}}(\mathbf{y},t)] = -\delta(\mathbf{x}-\mathbf{y})\,. \tag{7}$$

In this extended phase space the generator of the BRST symmetry is given by

$$Q = -\int d^3x \Big[C\Big(\nabla\cdot\boldsymbol{\pi} + e\psi^\dagger\psi\Big) + i\bar{\mathcal{P}}\pi_0 \Big]\,. \tag{8}$$

The striking property of Q is that $[Q, Q] = 0$ off-shell, with this at hand we can define a classical BRST cohomology for any gauge theory. It follows from $[Q, Q] = 0$ and the Jacobi identity for the Poisson brackets that $[[\mathcal{F}, Q], Q] = 0$ for any $\mathcal{F}$. From that we conclude that any BRST invariant quantity is defined modulo a Poisson bracket $[\mathcal{F}, Q]$. In our case we have that $[H_0, Q] = 0$ and so it is for the extended H:

$$H = H_0 + [Q, \Psi] \tag{9}$$

Here Ψ is an arbitrary fermionic functional of the extended phase space variables, that is chosen to give dynamics to all fields and to select a gauge slice. The extended phase space action is then

$$S_{BFV} = \int d^4x \Big(\dot{A}_\mu \pi^\mu + i\psi^\dagger\dot{\psi} + \dot{C}\mathcal{P} + \dot{\bar{C}}\bar{\mathcal{P}} \Big) - \int dt \Big(H_0 + [Q, \Psi] \Big)\,. \tag{10}$$

By construction this action is invariant under Q, that is, under the global transformations

$$\delta A_i = \partial_i C \qquad , \qquad \delta A_0 = i\bar{\mathcal{P}}\,, \tag{11}$$

$$\delta \pi_\mu = 0 \qquad , \qquad \delta\psi = -ieC\psi\,, \tag{12}$$

$$\delta C = 0 \qquad , \qquad \delta\bar{C} = i\pi_0\,, \tag{13}$$

$$\delta\mathcal{P} = \nabla\cdot\boldsymbol{\pi} + e\psi^\dagger\psi \qquad , \qquad \delta\bar{\mathcal{P}} = 0\,. \tag{14}$$

The Fradkin-Vilkovisky theorem [5] states that the path integral

$$Z_{BFV} = \int [dA_\mu]\,[d\pi_\mu]\,[dC]\,[d\bar{C}]\,[d\mathcal{P}]\,[d\bar{\mathcal{P}}]\,e^{iS_{BFV}}\,, \tag{15}$$

has no dependence on Ψ. For QED a common choice of Ψ is

$$\Psi = \int d^3x \Big[\mathcal{P}A_0 - i\bar{C}(\nabla\cdot\mathbf{A} - \frac{\xi}{2}\pi_0)\Big]\,, \tag{16}$$

where ξ is a real parameter that describes a set of gauges, e.g. for $\xi = 0$ we have the Landau gauge, for $\xi = 1$ the Feynman gauge, and for $\xi = \infty$ the unitary gauge. That choice of Ψ give us the action

$$\begin{aligned} S_{BFV} = \int d^4x \Big[& -\dot{\mathbf{A}}\cdot\boldsymbol{\pi} + i\psi^\dagger\dot{\psi} + \dot{C}\mathcal{P} + \dot{\bar{C}}\bar{\mathcal{P}} \\ & - \Big(\psi^\dagger(i\boldsymbol{\alpha}\cdot\mathbf{D} + \gamma_0 m)\psi + \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{4}F^{ij}F_{ij}\Big) \\ & - A_0(\nabla\cdot\boldsymbol{\pi} + e\psi^\dagger\psi) + \frac{\xi}{2}\pi_0^2 + \pi_0\partial_\mu A^\mu + i\mathcal{P}\bar{\mathcal{P}} - i\bar{C}\nabla^2 C\Big]\,. \end{aligned} \tag{17}$$

Performing the functional integrals over the momenta we recover the usual Faddeev-Popov action

$$S_{FP} = \int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + i\bar{C}\partial_\mu\partial^\mu C\Big]\,, \tag{18}$$

with its invariance under the BRST transformations

$$\delta A_\mu = \partial_\mu C \qquad , \qquad \delta\psi = -ieC\psi\,, \tag{19}$$

$$\delta C = 0 \qquad , \qquad \delta\bar{C} = -\frac{i}{\xi}\partial_\mu A^\mu\,. \tag{20}$$

Recently Lavelle and McMullan [2] discovered that the Faddeev-Popov action is also invariant under the nonlocal variations:

$$\delta^\perp A_i = i\frac{\partial_i\partial_0}{\nabla^2}\bar{C} \qquad , \qquad \delta^\perp A_0 = i\bar{C}\,, \tag{21}$$

$$\delta^\perp\psi = e\Big(\frac{\partial_0}{\nabla^2}\bar{C}\Big)\psi \qquad , \qquad \delta^\perp C = A_0 - \frac{\partial_i\partial_0}{\nabla^2}A_i + \frac{e}{\nabla^2}\psi^\dagger\psi\,, \tag{22}$$

$$\delta^\perp\bar{C} = 0\,. \tag{23}$$

This was found in the search for a symmetry that decreases the ghost number by one and that obeys $\delta^{\perp}(\partial_\mu A^\mu) = 0$, i.e., leaves the gauge fixing condition invariant. And also with $[Q, Q^{\perp}] \neq 0$. This symmetry was discovered in the configuration space. The question we pose is: How does it arise in the BFV phase space? In the phase space formulation we have the freedom of performing canonical transformations, in such a way that any two BRST generators are related by such transformations [6]. We propose now the following canonical transformation in the ghost sector:

$$C' = \frac{1}{\nabla^2}\mathcal{P} \qquad , \qquad \mathcal{P}' = \nabla^2 C\,, \tag{24}$$

$$\bar{C}' = \bar{\mathcal{P}} \qquad , \qquad \bar{\mathcal{P}}' = \bar{C} \tag{25}$$

It is easy to see that the BFV Lagrangian is form invariant under these replacements, so that dropping the primes we have the new BRST charge, that we call $Q^{\perp}$, being

$$Q^{\perp} = -\int d^3x \Big[\frac{1}{\nabla^2}\mathcal{P}\Big(\nabla\cdot\boldsymbol{\pi} + e\psi^\dagger\psi\Big) + i\bar{C}\pi_0\Big]\,. \tag{26}$$

The action of this charge on the extended phase space is

$$\delta^{\perp}A_i = \frac{\partial_i}{\nabla^2}\mathcal{P} \qquad , \qquad \delta^{\perp}A_0 = i\bar{C}\,, \tag{27}$$

$$\delta^{\perp}\pi_\mu = 0 \qquad , \qquad \delta^{\perp}\psi = -ie\frac{1}{\nabla^2}\mathcal{P}\psi\,, \tag{28}$$

$$\delta^{\perp}C = \frac{1}{\nabla^2}(\nabla\cdot\boldsymbol{\pi} + e\psi^\dagger\psi) \qquad , \qquad \delta^{\perp}\bar{C} = 0\,, \tag{29}$$

$$\delta^{\perp}\mathcal{P} = 0 \qquad , \qquad \delta^{\perp}\bar{\mathcal{P}} = i\pi_0\,. \tag{30}$$

It is straightforward to see that on integration over the momenta these transformations reduce to the ones found in [2]. Among the huge number of possible canonical transformations, this one meets the requirements of changing the ghost number of Q to minus one and that after integration over the momenta gives $\delta^{\perp}(\partial_\mu A^\mu) = 0$ off-shell. To understand this last property recall that in the extended phase space we have $\delta(\partial_\mu A^\mu) = i\dot{\bar{\mathcal{P}}} - \nabla^2 C$ which in configuration space translates into $\delta(\partial_\mu A^\mu) = \partial_\mu\partial^\mu C$, but this is just the classical equation of motion of C. What the above proposed canonical transformation performs is a swap of the ghost Hamilton equations of motion, that is, $\delta^{\perp}(\partial_\mu A^\mu) = i\dot{\bar{C}} - \mathcal{P}$, which after integration

over the momenta gives zero, turning the variation of the gauge fixing condition from null on-shell to null off-shell .

Acknowledgments

The authors are grateful to C. A. Linhares for reading the manuscript. P. Gaete thanks the Instituto de Física of UFRJ for the hospitality. This work was partially supported by CNPq (Brazilian Research Council).

REFERENCES

[1] C. Bechi, A. Rouet and R. Stora, Phys. Lett. B **52** , 344 (1974); Ann. Phys. **98**, 287 (1976); I.V. Tyutin, Lebedev preprint FIAN 39 (1975), unpublished.

[2] M. Lavelle and D. McMullan, Phys. Rev. Lett. **71**, 3758 (1993).

[3] Z. Tang and D. Finkelstein, Phys. Rev. Lett. **73**, 3055 (1994).

[4] H. Yang and B.-H. Lee, preprint hep-th 9502081.

[5] E.S. Fradkin and G.A. Vilkovsky, Phys. Lett. B **55**, 224 (1975); I.A. Batalin and G.A. Vilkovsky, Phys. Lett. B **69** , 309 (1977).

[6] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton Univ. Press, Princeton, 1992); M. Henneaux, Phys. Rep. 126, 1 (1985).